Problem 1.28

Prove that the curl of a gradient is always zero. *Check* it for function (b) in Prob. 1.11.

Solution

Evaluate the curl of a gradient explicitly.

$$\begin{aligned}
\nabla \times (\nabla f) &= \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) \times \left[\left(\sum_{j=1}^{3} \boldsymbol{\delta}_j \frac{\partial}{\partial x_j}\right) f\right] \\
&= \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i \frac{\partial}{\partial x_i}\right) \times \left(\sum_{j=1}^{3} \boldsymbol{\delta}_j \frac{\partial f}{\partial x_j}\right) \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3} (\boldsymbol{\delta}_i \times \boldsymbol{\delta}_j) \frac{\partial}{\partial x_i}\frac{\partial f}{\partial x_j} \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \boldsymbol{\delta}_k \varepsilon_{ijk} \frac{\partial^2 f}{\partial x_i \partial x_j} \\
&= \sum_{j=1}^{3}\sum_{i=1}^{3}\sum_{k=1}^{3} \boldsymbol{\delta}_k \varepsilon_{jik} \frac{\partial^2 f}{\partial x_j \partial x_i} \qquad \text{(Let } i \text{ be } j \text{ and let } j \text{ be } i.) \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \boldsymbol{\delta}_k \varepsilon_{jik} \frac{\partial^2 f}{\partial x_j \partial x_i} \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \boldsymbol{\delta}_k \varepsilon_{jik} \frac{\partial^2 f}{\partial x_i \partial x_j} \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \boldsymbol{\delta}_k (-\varepsilon_{ijk}) \frac{\partial^2 f}{\partial x_i \partial x_j} \\
&= -\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \boldsymbol{\delta}_k \varepsilon_{ijk} \frac{\partial^2 f}{\partial x_i \partial x_j} \\
&= \mathbf{0}
\end{aligned}$$

Then verify it for $f(x,y,z) = x^2y^3z^4$.

$$
\begin{aligned}
\nabla \times (\nabla f) &= \nabla \times \left[\hat{\mathbf{x}}\frac{\partial}{\partial x}(x^2y^3z^4) + \hat{\mathbf{y}}\frac{\partial}{\partial y}(x^2y^3z^4) + \hat{\mathbf{z}}\frac{\partial}{\partial z}(x^2y^3z^4)\right] \\
&= \nabla \times \left[\hat{\mathbf{x}}(2xy^3z^4) + \hat{\mathbf{y}}(3x^2y^2z^4) + \hat{\mathbf{z}}(4x^2y^3z^3)\right] \\
&= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy^3z^4 & 3x^2y^2z^4 & 4x^2y^3z^3 \end{vmatrix} \\
&= \hat{\mathbf{x}}\left[\frac{\partial}{\partial y}(4x^2y^3z^3) - \frac{\partial}{\partial z}(3x^2y^2z^4)\right] - \hat{\mathbf{y}}\left[\frac{\partial}{\partial x}(4x^2y^3z^3) - \frac{\partial}{\partial z}(2xy^3z^4)\right] + \hat{\mathbf{z}}\left[\frac{\partial}{\partial x}(3x^2y^2z^4) - \frac{\partial}{\partial y}(2xy^3z^4)\right] \\
&= \hat{\mathbf{x}}\left[(12x^2y^2z^3) - (12x^2y^2z^3)\right] - \hat{\mathbf{y}}\left[(8xy^3z^3) - (8xy^3z^3)\right] + \hat{\mathbf{z}}\left[(6xy^2z^4) - (6xy^2z^4)\right] \\
&= \hat{\mathbf{x}}(0) - \hat{\mathbf{y}}(0) + \hat{\mathbf{z}}(0) \\
&= \mathbf{0}
\end{aligned}
$$